

THE FRENET-SERRET APPARATUS AND LOCAL SINGULAR VALUE DECOMPOSITION OF CURVES IN \mathbb{R}^n

ROBERT ARN, BRUCE DRAPER, MICHAEL KIRBY, AND CHRIS PETERSON

We would like to dedicate this paper to David Broomhead (1950-2014), a colleague and friend, who conceived many beautiful geometric applications of the singular value decomposition.

ABSTRACT. Let $\gamma : I \rightarrow \mathbb{R}^n$ be a parametric curve of class C^{n+1} , regular of order n . The Frenet-Serret apparatus of γ at $\gamma(t)$ consists of a frame $e_1(t), \dots, e_n(t)$ and generalized curvature values $\kappa_1(t), \dots, \kappa_{n-1}(t)$. Associated with each point of γ there are also local singular vectors $u_1(t), \dots, u_n(t)$ and local singular values $\sigma_1(t), \dots, \sigma_n(t)$. This local information is obtained by considering a limit, as ϵ goes to zero, of covariance matrices defined along γ within an ϵ -ball centered at $\gamma(t)$. We prove that for each $t \in I$, the Frenet-Serret frame and the local singular vectors agree at $\gamma(t)$ and that for $n \leq 6$, the values of the curvature functions at t can be expressed as a fixed multiple of a ratio of local singular values at t . More precisely, we show that if $\gamma(t) \subset \mathbb{R}^n$ with $n \leq 6$ then, for each i between 2 and n , $\kappa_{i-1}(t) = \sqrt{a_{i-1}} \frac{\sigma_i(t)}{\sigma_1(t)\sigma_{i-1}(t)}$ with $a_{i-1} = \left(\frac{i}{i+(-1)^i}\right)^2 \frac{4i^2-1}{3}$.

1. INTRODUCTION

Consider an interval $I \subset \mathbb{R}$ and a vector valued function $\gamma : I \rightarrow \mathbb{R}^n$. If γ is k times differentiable, with continuous derivatives, then γ is said to be a parametric curve of class C^k . Let $\gamma^{(k)}$ denote the k^{th} derivative of γ . If for each $t \in I$, the set of vectors $\{\gamma^{(1)}(t), \gamma^{(2)}(t), \dots, \gamma^{(r)}(t)\}$ are linearly independent in \mathbb{R}^n , then γ is said to be regular of order r . If $\|\gamma'(t)\| = 1$ for each $t \in I$ then γ is said to be parameterized by arc length.

Let $\gamma : I \rightarrow \mathbb{R}^n$ be a parametric curve of class C^{n+1} , regular of order n , parameterized by arc length. At any point $\gamma(t) \in \gamma$, the Frenet-Serret frame is determined by applying the Gram-Schmidt process to the vectors $\gamma^{(1)}(t), \gamma^{(2)}(t), \dots, \gamma^{(n)}(t)$. Thus the Frenet-Serret frame at $\gamma(t)$ is the ordered sequence of orthonormal vectors $e_1(t), e_2(t), \dots, e_n(t)$ where

$$e_i(t) = \frac{\tilde{e}_i(t)}{\|\tilde{e}_i(t)\|} \quad \text{with} \quad \tilde{e}_i(t) = \gamma^{(i)}(t) - \sum_{k=1}^{i-1} \langle \gamma^{(i)}(t), e_k(t) \rangle e_k(t) \quad \text{for } 1 \leq i \leq n$$

The generalized curvature functions of γ are defined by

$$\kappa_i(t) = \langle e'_i(t), e_{i+1}(t) \rangle \quad \text{for } 1 \leq i \leq n-1.$$

With this definition, $\kappa_i(t) > 0$ for all i . The frame functions $e_1(t), e_2(t), \dots, e_n(t)$ together with the generalized curvature functions $\kappa_1(t), \dots, \kappa_{n-1}(t)$ is called the

Frenet-Serret apparatus of γ . The Frenet-Serret apparatus of a curve characterizes the curve up to translation.

By the definition of the $e_i(t)$, we have

$$e_i(t) \in \text{span}\{\gamma^{(1)}(t), \dots, \gamma^{(i)}(t)\} \quad \text{for } i = 1, \dots, n-1.$$

Thus,

$$e'_i(t) \in \text{span}\{\gamma^{(1)}(t), \dots, \gamma^{(i+1)}(t)\} = \text{span}\{e_1(t), \dots, e_{i+1}(t)\}.$$

As a consequence,

$$\langle e'_i(t), e_j(t) \rangle = 0 \quad \text{whenever } j \geq i+2.$$

If we differentiate the expression $\langle e_i(t), e_i(t) \rangle = 1$ then we obtain

$$\langle e'_i(t), e_i(t) \rangle + \langle e_i(t), e'_i(t) \rangle = 0$$

from which we can conclude that

$$\langle e'_i(t), e_i(t) \rangle = 0 \quad \text{for } 1 \leq i \leq n.$$

In a similar manner, if $i \neq j$ then we differentiate the expression $\langle e_i(t), e_j(t) \rangle = 0$ to obtain

$$\langle e'_i(t), e_j(t) \rangle + \langle e_i(t), e'_j(t) \rangle = 0$$

from which we can conclude that

$$\langle e'_i(t), e_j(t) \rangle = -\langle e'_j(t), e_i(t) \rangle.$$

Let E denote the orthonormal matrix whose columns are $e_1(t), \dots, e_n(t)$. The above formulas show that $E^T E' = K$ with K a block diagonal skew symmetric matrix. Since E is orthonormal (thus $EE^T = I$), we can multiply on the left by E to arrive at the expression $E' = EK$. Recalling that $\kappa_i(t) = \langle e'_i(t), e_{i+1}(t) \rangle$ we can express K as:

$$K = \begin{pmatrix} 0 & \kappa_1(t) & 0 & 0 & 0 \\ -\kappa_1(t) & 0 & \kappa_2(t) & 0 & 0 \\ 0 & -\kappa_2(t) & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 0 & \kappa_{n-1}(t) \\ 0 & 0 & 0 & -\kappa_{n-1}(t) & 0 \end{pmatrix}$$

If the generalized curvature functions $\kappa_1(t), \dots, \kappa_{n-1}(t)$ in the matrix K are constant, then the solution to the differential equation, $E' = EK$, can be shown to be (up to translation) of the form

$$(1.1) \quad \gamma_e(t) = \begin{bmatrix} a_1 \cos(\alpha_1 t) \\ a_1 \sin(\alpha_1 t) \\ \vdots \\ a_k \cos(\alpha_k t) \\ a_k \sin(\alpha_k t) \end{bmatrix} \quad \text{or} \quad \gamma_o(t) = \begin{bmatrix} a_1 \cos(\alpha_1 t) \\ a_1 \sin(\alpha_1 t) \\ \vdots \\ a_k \cos(\alpha_k t) \\ a_k \sin(\alpha_k t) \\ bt \end{bmatrix}$$

with the first equation, $\gamma_e(t)$, covering the case when n is even with $k = n/2$ and the second equation covering the case when n is odd with $k = (n-1)/2$ [8].

2. LOCAL APPROXIMATION

Consider a curve $\gamma(t)$ in \mathbb{R}^n . Recall that if $\gamma(t)$ is parameterized by arc length then $\gamma(t)$ is a solution to the differential equation $E' = EK$. We would like to understand the associated frame $e_1(t), \dots, e_n(t)$ and curvature functions $\kappa_1(t), \dots, \kappa_{n-1}(t)$ from a different point of view. Specifically, consider points on the curve within an ϵ -ball centered at a point $s_0 = \gamma(t_0)$. The tangent line at s_0 is approximated by taking the span of two points on $\gamma(t)$ in an ϵ -ball centered at s_0 while the *osculating* plane at s_0 is approximated by taking the span of three points on $\gamma(t)$ in an ϵ -ball centered at s_0 . However, points on the curve in a small ϵ -ball are nearly linear. The value of $\kappa_1(t_0)$ can be seen as a measure of the failure of the linearity of such points. In a similar manner, the value of the second curvature function, $\kappa_2(t_0)$ is a measure of the failure of planarity of points in an ϵ -ball on the curve. This point of view will be considered more closely in the next section through the local singular value decomposition. In order to make this connection, it is helpful to replace the curve with an idealized version which agrees, to high order, with the curve at $\gamma(t_0)$.

2.1. Local approximation of curves in \mathbb{R}^3 and \mathbb{R}^4 . Consider a curve $\gamma(t)$ in \mathbb{R}^3 . The helix of best fit to γ at $\gamma(t_0)$ is the solution to the differential equation $E' = EK_{t_0}$ where K_{t_0} denotes the curvature matrix K evaluated at t_0 . Thus the curvature functions for the helix will be constants $\kappa_1 = \kappa_1(t_0)$ and $\kappa_2 = \kappa_2(t_0)$. The general solution, $g(t)$, to the differential equation, $E' = EK_{t_0}$, has the form

$$g(t) = (a \cos(\alpha t), a \sin(\alpha t), bt) + Constant.$$

The helix of best fit to $\gamma(t)$ at $\gamma(t_0)$ is given by

$$h(t) = g(t) - g(t_0) + \gamma(t_0).$$

If $\|\gamma^{(1)}(t_0)\| = 1$ then we get the condition that

$$(2.1) \quad a^2 \alpha^2 + b^2 = 1$$

The relationship between the curvature functions of the helix and the parameters a, b, α is:

$$(2.2) \quad \kappa_1^2 = a^2 \alpha^4$$

$$(2.3) \quad \kappa_2^2 = b^2 \alpha^2$$

Following this pattern, if we solve the differential equation $E' = EK_{t_0}$ for a curve $\gamma(t)$ in \mathbb{R}^4 then we obtain a toroidal curve of best fit at $\gamma(t_0)$ of the form

$$h(t) = g(t) - g(t_0) + \gamma(t_0)$$

where

$$g(t) = (a \cos(\alpha t), a \sin(\alpha t), b \cos(\beta t), b \sin(\beta t)) + Constant.$$

We can relate a, b, α, β to the curvature functions as

$$(2.4) \quad \kappa_1^2 = a^2 \alpha^4 + b^2 \beta^4$$

$$(2.5) \quad \kappa_1^2 \kappa_2^2 = a^2 \alpha^6 + b^2 \beta^6 - \kappa_1^4$$

$$(2.6) \quad \kappa_1^2 \kappa_2^2 \kappa_3^2 = a^2 \alpha^8 + b^2 \beta^8 - \kappa_1^2 (\kappa_1^2 + \kappa_2^2)^2$$

where again we have assumed that the curve is parameterized by arc length so

$$(2.7) \quad a^2 \alpha^2 + b^2 \beta^2 = 1$$

These equations are derived for $\kappa_1, \kappa_2, \kappa_3$ in [8]. Next we give the corresponding equations for curves in \mathbb{R}^5 and \mathbb{R}^6 . The derivation is straightforward but tedious.

2.2. Curvature relations $n = 5$. If we solve the differential equation $E' = EK_{t_0}$ for a curve $\gamma(t)$ in \mathbb{R}^5 then we obtain a curve of best fit at $\gamma(t_0)$ of the form

$$h(t) = g(t) - g(t_0) + \gamma(t_0)$$

where

$$g(t) = (a \cos(\alpha t), a \sin(\alpha t), b \cos(\beta t), b \sin(\beta t), ct) + \text{Constant}.$$

We can relate a, b, c, α, β to the curvature functions as

$$\begin{aligned} 1 &= a^2 \alpha^2 + b^2 \beta^2 + c^2 \\ \kappa_1^2 &= a^2 \alpha^4 + b^2 \beta^4 \\ \kappa_1^2 \kappa_2^2 &= a^2 \alpha^6 + b^2 \beta^6 - \kappa_1^4 \\ \kappa_1^2 \kappa_2^2 \kappa_3^2 &= a^2 \alpha^8 + b^2 \beta^8 - \kappa_1^2 (\kappa_1^2 + \kappa_2^2)^2 \\ \kappa_1^2 \kappa_2^2 \kappa_3^2 \kappa_4^2 &= a^2 \alpha^{10} + b^2 \beta^{10} - \kappa_1^2 ((\kappa_1^2 + \kappa_2^2 + \kappa_3^2)(\kappa_2^2 + \kappa_3^2) + \kappa_2^2 \kappa_3^4) \end{aligned}$$

2.3. Curvature relations $n = 6$. If we solve the differential equation $E' = EK_{t_0}$ for a curve $\gamma(t)$ in \mathbb{R}^6 then we obtain a curve of best fit at $\gamma(t_0)$ of the form

$$h(t) = g(t) - g(t_0) + \gamma(t_0)$$

where

$$g(t) = (a \cos(\alpha t), a \sin(\alpha t), b \cos(\beta t), b \sin(\beta t), c \cos(\delta t), c \sin(\delta t)) + \text{Constant}.$$

Letting $F_k = a^2 \alpha^k + b^2 \beta^k + c^2 \delta^k$, we can relate $a, b, c, \alpha, \beta, \delta$ to the curvature functions as

$$\begin{aligned} 1 &= F_2 \\ \kappa_1^2 &= F_4 \\ \kappa_1^2 \kappa_2^2 &= F_6 - \kappa_1^4 \\ \kappa_1^2 \kappa_2^2 \kappa_3^2 &= F_8 - \kappa_1^2 (\kappa_1^2 + \kappa_2^2)^2 \\ \kappa_1^2 \kappa_2^2 \kappa_3^2 \kappa_4^2 &= F_{10} - \kappa_1^2 ((\kappa_1^2 + \kappa_2^2 + \kappa_3^2)(\kappa_2^2 + \kappa_3^2) + \kappa_2^2 \kappa_3^4) \\ \kappa_1^2 \kappa_2^2 \kappa_3^2 \kappa_4^2 \kappa_5^2 &= F_{12} - F_{10} (\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) + F_8 (\kappa_1^2 \kappa_3^2 + \kappa_4^2 \kappa_1^2 + \kappa_4^2 \kappa_2^2) \end{aligned}$$

3. THE LOCAL SINGULAR VALUE DECOMPOSITION

Recall that at each point $\gamma(t) \in \gamma$, the Frenet-Serret frame is determined by applying the Gram-Schmidt process to the vectors $\gamma^{(1)}(t), \gamma^{(2)}(t), \dots, \gamma^{(n)}(t)$ (where $\gamma^{(k)}(t)$ denotes the k^{th} derivative of γ evaluated at t). We denote this ordered orthonormal basis $e_1(t), \dots, e_n(t)$ and let E denote the orthonormal matrix whose columns are the $e_i(t)$. The main intuition behind a local singular value analysis is to exploit the idea that the Frenet-Serret frame may be viewed as finding the

subspace of best fit at a point on the curve. We consider the canonical solution of the Frenet-Serret formula where κ_i is assumed to be constant, i.e., the solutions to $E' = EK$ given by Equation (1.1) where K is constant. We use an integral formulation of the singular value decomposition, often referred to as the Karhunen-Loève transformation, at a given point on the curve. We then use a Taylor series approximation for $\gamma(t)$ to determine particular eigenvalues of the Karhunen-Loève transformation in the ϵ -ball. These relationships can be combined with the relationships between the curvature constants and the curve parameters to determine a formula for computing κ_i locally from the singular values of the Karhunen-Loève transformation.

3.1. Formulation. Broomhead et al showed that the *local* singular value decomposition could be used to compute the topological dimension of a manifold from sampled points lying on the manifold [2]. This provided a powerful tool for many applications that involved modeling data on manifolds. The original setting of [2] concerned the reconstruction of a manifold, via Takens' theorem, from scalar valued time series statistics of a dynamical system on the manifold. The local singular value decomposition is also useful for applying manifold learning algorithms for geometric data analysis, e.g., local models such as charts [6], or global models based on Whitney's embedding theorem [4]. A more detailed discussion may be found in [7, 9].

Following [3, 2], the *mean centered* covariance matrix of $\gamma(t)$ at t is the matrix

$$\overline{C}_\epsilon(t) = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} (\gamma(s) - \overline{\gamma}_\epsilon(t))(\gamma(s) - \overline{\gamma}_\epsilon(t))^T ds$$

where

$$\overline{\gamma}_\epsilon(t) = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} \gamma(s) ds$$

However, we will consider the closely related *on the curve* covariance matrix

$$C_\epsilon(t) = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} (\gamma(s) - \gamma(t))(\gamma(s) - \gamma(t))^T ds$$

By the singular value decomposition, we have a factorization

$$C_\epsilon(t) = U_\epsilon(t) \Sigma_\epsilon(t) U_\epsilon^T(t)$$

where we assume that the diagonal elements in $\Sigma_\epsilon(t)$ are in monotone decreasing order. We call the columns of $U_\epsilon(t)$ the singular vectors of $C_\epsilon(t)$. Note that such singular vectors are only defined up to a factor of ± 1 . Let $U(t) = \lim_{\epsilon \rightarrow 0} U_\epsilon(t)$. The columns of $U(t)$, written $u_1(t), \dots, u_n(t)$, are called the local singular vectors at $\gamma(t)$. In a similar manner, one can define the local singular vectors $\overline{u}_1(t), \dots, \overline{u}_n(t)$ at $\gamma(t)$ by considering the limiting behavior of the singular vectors in the singular value decomposition of $\overline{C}_\epsilon(t)$ as ϵ tends towards zero.

Theorem 3.1. *Let $\gamma : I \rightarrow \mathbb{R}^n$ be a parametric curve of class C^{n+1} , regular of order n . Let $e_1(t), \dots, e_n(t)$ denote the Frenet-Serret frame at $\gamma(t)$. Let $u_1(t), \dots, u_n(t)$ denote the local singular vectors at $\gamma(t)$. Then for $i = 1, \dots, n$, $e_i(t) = \pm u_i(t)$.*

Proof. Let $\Gamma(t)$ denote the matrix whose columns are $\gamma^{(1)}(t), \dots, \gamma^{(n)}(t)$. The Frenet-Serret frame, $e_1(t), \dots, e_n(t)$, is obtained by applying the Gram-Schmidt process to the columns of $\Gamma(t)$. Thus $e_i(t)$ is a unit vector orthogonal to the span of $\gamma^{(1)}(t), \dots, \gamma^{(i-1)}(t)$ but lying within the span of $\gamma^{(1)}(t), \dots, \gamma^{(i)}(t)$. Let \mathbf{v} be

the $n \times 1$ vector whose k^{th} component is $(s - t)^k/k!$. Then $\Gamma(t)\mathbf{v}$ is the n^{th} order Taylor series expansion for $\gamma(s) - \gamma(t)$ at t . Replacing $\gamma(s) - \gamma(t)$ with its Taylor series expansion leads to the n^{th} order approximation

$$C_\epsilon(t) = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} (\gamma(s) - \gamma(t))(\gamma(s) - \gamma(t))^T ds \approx \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} (\Gamma(t)\mathbf{v})(\Gamma(t)\mathbf{v})^T ds$$

We rewrite this as

$$\Gamma(t) \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} \mathbf{v}\mathbf{v}^T ds \Gamma(t)^T = \Gamma(t) \mathcal{E} \Gamma(t)^T$$

By the definition of \mathcal{E} , we compute that

$$\mathcal{E}_{i,j} = \frac{\epsilon^{i+j}}{i!j!(i+j+1)} \quad \text{if } i+j \text{ is even} \quad \text{and} \quad \mathcal{E}_{i,j} = 0 \quad \text{if } i+j \text{ is odd.}$$

We can express $\Gamma(t) \mathcal{E} \Gamma(t)^T$ in terms of the columns of $\Gamma(t)$ and the entries of \mathcal{E} as

$$\frac{\epsilon^2}{3}(c_1 c_1^T) + \frac{\epsilon^4}{5}\left(\frac{1}{6}c_1 c_3^T + \frac{1}{4}c_2 c_2^T + \frac{1}{6}c_3 c_1^T\right) + \dots + \frac{\epsilon^{2k}}{2k+1} \sum_{i=1}^{2k-1} \frac{1}{i!(2k-i)!} c_i c_{2k-i}^T + \dots$$

where $c_i = \gamma^{(i)}(t)$. As ϵ tends towards zero, this expression behaves more and more like the rank one matrix $\frac{\epsilon^2}{3}c_1 c_1^T$. Noting that $c_1 = \gamma^{(1)}(t)$, thus is a multiple of $e_1(t)$, we get $u_1(t) = \pm e_1(t)$. Let $P_1 = I - e_1(t)e_1(t)^T$. Pre and post multiplying $\Gamma(t) \mathcal{E} \Gamma(t)^T$ with P_1 deflates away all terms involving c_1 . More precisely,

$$P_1 \Gamma(t) \mathcal{E} \Gamma(t)^T P_1 = \frac{\epsilon^4}{5}\left(\frac{1}{4}P_1 c_2 c_2^T P_1\right) + \dots + \frac{\epsilon^{2k}}{2k+1} \sum_{i=2}^{2k-2} \frac{1}{i!(2k-i)!} P_1 c_i c_{2k-i}^T P_1 + \dots$$

As ϵ tends towards zero, this deflated matrix behaves more and more like the rank one matrix $\frac{\epsilon^4}{5}\left(\frac{1}{4}P_1 c_2 c_2^T P_1\right)$. Noting that $P_1 c_2 = P_1 \gamma^{(2)}(t)$, we see that $P_1 c_2$ is orthogonal to $\gamma^{(1)}$ and is in the span of $\gamma^{(1)}, \gamma^{(2)}$ thus is a multiple of $e_2(t)$. This leads to $u_2(t) = \pm e_2(t)$. We now pre and post multiply $P_1 \Gamma(t) \mathcal{E} \Gamma(t)^T P_1$ with $P_2 = I - e_2(t)e_2(t)^T$. Note that since $e_1(t)$ is orthogonal to $e_2(t)$, we have $P_2 P_1 = I - e_1(t)e_1(t)^T - e_2(t)e_2(t)^T$. As ϵ tends towards zero, this doubly deflated matrix behaves more and more like the rank one matrix $\frac{\epsilon^6}{7}\left(\frac{1}{36}P_2 P_1 c_3 c_3^T P_1 P_2\right)$. Noting that $P_2 P_1 c_3 = P_2 P_1 \gamma^{(3)}(t)$, we see that $P_2 P_1 c_3$ is orthogonal to the span of $\gamma^{(1)}, \gamma^{(2)}$ but in the span of $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$ thus is a multiple of $e_3(t)$. This leads to $u_3(t) = \pm e_3(t)$. Continuing to deflate away previously found singular vectors, we obtain the relationship $e_i(t) = \pm u_i(t)$ for all i . Note that for this to work, $\mathcal{E}_{i,i}$ must be non-zero and $P_i P_{i-1} \dots P_1 \gamma^{(i+1)}(t)$ must be non-zero for each i . These conditions are satisfied since $\mathcal{E}_{i,i} = \frac{\epsilon^{2i}}{(2i+1)i!i!}$ and γ is regular of order n thus $\gamma^{(1)}(t), \dots, \gamma^{(n)}(t)$ are linearly independent. \square

The previous theorem considered the relationship between the local singular vectors of a curve and the Frenet-Serret frame of a curve. We now consider the relationship between the local singular values of a curve and values of the curvature functions. More precisely, in the singular value decomposition

$$C_\epsilon(t) = U_\epsilon(t) \Sigma_\epsilon(t) U_\epsilon^T(t)$$

we considered the limiting behavior of $U_\epsilon(t)$, as ϵ tends towards zero, in order to obtain the local singular vectors. We now consider the limiting behavior of $\Sigma_\epsilon(t)$

as ϵ tends towards zero. Note that the entries of $\Sigma_\epsilon(t)$ are the eigenvalues of $C_\epsilon(t)$ and that they tend towards zero as ϵ tends towards zero. Let $\lambda_{i,\epsilon}(t)$ denote the i^{th} diagonal entry of $\Sigma_\epsilon(t)$. We show that for some constant c_i , we can write

$$\lambda_{i,\epsilon}(t) = c_i \epsilon^{2i} + O(\epsilon^{2i+2})$$

The local singular values of $\gamma(t)$ are then defined as $\sigma_i(t) = \sqrt{c_i} \epsilon^i$.

In Section 2, we have explicitly expressed the curvature, for curves with constant curvature functions, in terms of the parameters of the curves. We now express the leading terms of the eigenvalues $\lambda_{i,\epsilon}(t)$ in terms of the parameters of the curves. This allows us to derive a relationship of the form

$$\kappa_i^2(t) = a_i \lim_{\epsilon \rightarrow 0} \frac{\lambda_{i+1,\epsilon}(t)}{\lambda_{1,\epsilon}(t) \lambda_{i,\epsilon}(t)}$$

where a_i is a constant with known value. From this we obtain

$$\kappa_i(t) = \sqrt{a_i} \frac{\sigma_{i+1}(t)}{\sigma_1(t) \sigma_i(t)}$$

3.2. Two dimensions. Consider a two dimensional curve with constant curvature $\kappa_1 = 1/a$. This will be a circle of radius a . Up to translation, its parameterized form is $\gamma(s) = (a \cos(\alpha s), a \sin(\alpha s))$. If we assume that the circle is parameterized by arc length then we obtain the constraint $a^2 \alpha^2 = 1$. The components of the covariance matrix $C_\epsilon(0)$ are:

$$C_{11} = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} (a \cos(\alpha s) - a)^2 ds$$

$$C_{22} = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} a^2 \sin^2(\alpha s) ds$$

with

$$C_{12} = C_{21} = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} (a \cos(\alpha s) - a) \sin(s) ds = 0$$

since the integrand is an odd function.

We follow the usual convention of ordering the eigenvalues by decreasing magnitude so

$$\lambda_{1,\epsilon}(0) = \frac{1}{3} a^2 \alpha^2 \epsilon^2 + O(\epsilon^4)$$

$$\lambda_{2,\epsilon}(0) = \frac{1}{20} a^2 \alpha^4 \epsilon^4 + O(\epsilon^6)$$

$$(3.1) \quad \lim_{\epsilon \rightarrow 0} \frac{\lambda_{2,\epsilon}(0)}{\lambda_{1,\epsilon}^2(0)} = \frac{9}{20a^2}$$

Given that the curvature $\kappa_1 = 1/a$, we obtain the following expression for κ_1 in terms of the local singular values of the circle:

$$(3.2) \quad \kappa_1 = \sqrt{\frac{20}{9} \frac{\sigma_2}{\sigma_1^2}} = \frac{\sqrt{20}}{3} \frac{\sigma_2}{\sigma_1^2}$$

3.3. Three dimensions. Here we consider curves in \mathbb{R}^3 with constant κ_1, κ_2 . Up to translation, such a curve will have the form

$$\gamma(s) = (a \cos(\alpha s), a \sin(\alpha s), bs)$$

Assuming the curve is parameterized by arc length we have $a^2\alpha^2 + b^2 = 1$. The covariance matrix, $C_\epsilon(t)$, is a 3×3 matrix with eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{1}{3}\epsilon^2 + O(\epsilon^4) \\ \lambda_2 &= \frac{1}{20}a^2\alpha^4\epsilon^4 + O(\epsilon^6) \\ \lambda_3 &= \frac{1}{1575}a^2\alpha^6b^2\epsilon^6 + O(\epsilon^8) \end{aligned}$$

Recalling from Section 2 the equations for κ_1, κ_2 in terms of the parameters a, α, b , we obtain

$$(3.3) \quad \kappa_1^2 = \frac{20}{9} \lim_{\epsilon \rightarrow 0} \frac{\lambda_{2,\epsilon}(t)}{\lambda_{1,\epsilon}^2(t)} \quad \kappa_2^2 = \frac{105}{4} \lim_{\epsilon \rightarrow 0} \frac{\lambda_{3,\epsilon}(t)}{\lambda_{1,\epsilon}(t)\lambda_{2,\epsilon}(t)}$$

This leads to the expression of κ_1, κ_2 in terms of the singular values as:

$$\kappa_1 = \frac{\sqrt{20}}{3} \frac{\sigma_2}{\sigma_1^2} \quad \text{and} \quad \kappa_2 = \frac{\sqrt{105}}{2} \frac{\sigma_3}{\sigma_1\sigma_2}$$

3.4. Four Dimensions. Here we consider curves in \mathbb{R}^4 with constant $\kappa_1, \kappa_2, \kappa_3$. Up to translation, such a curve will have the form

$$\gamma(t) = (a \cos(\alpha t), a \sin(\alpha t), b \cos(\beta t), b \sin(\beta t))$$

This leads to the following formulas:

$$\begin{aligned} 1 &= a^2\alpha^2 + b^2\beta^2 \\ \lambda_1 &= \frac{1}{3}\epsilon^2 + O(\epsilon^4) \\ \lambda_2 &= \frac{1}{20}a^2\alpha^4 + b^2\beta^4\epsilon^4 + O(\epsilon^6) \\ \lambda_3 &= \frac{1}{1575}a^2b^2\alpha^2\beta^2(\alpha^2 - \beta^2)^2\epsilon^6 + O(\epsilon^8) \\ \lambda_4 &= \frac{1}{63504} \frac{a^2b^2\alpha^4\beta^4(\alpha^2 - \beta^2)^2}{a^2\alpha^4 + b^2\beta^4} \epsilon^8 + O(\epsilon^{10}) \end{aligned}$$

Using elimination theory we establish the following representations of the κ_i in terms of the local singular values:

$$\kappa_1 = \frac{\sqrt{20}}{3} \frac{\sigma_2}{\sigma_1^2}, \quad \kappa_2 = \frac{\sqrt{105}}{2} \frac{\sigma_3}{\sigma_1\sigma_2}, \quad \kappa_3 = \frac{\sqrt{336}}{5} \frac{\sigma_4}{\sigma_1\sigma_3}$$

3.5. Patterns in higher dimensions. Given that many of the entries of $C_\epsilon(0)$ are odd functions, the covariance matrix has a special structure with many zero entries. For instance, the structure of the covariance matrix for $n = 6$ is

$$\begin{bmatrix} C_{11} & 0 & C_{13} & 0 & C_{15} & 0 \\ 0 & C_{22} & 0 & C_{24} & 0 & C_{26} \\ C_{31} & 0 & C_{33} & 0 & C_{35} & 0 \\ 0 & C_{42} & 0 & C_{44} & 0 & C_{46} \\ C_{51} & 0 & C_{53} & 0 & C_{55} & 0 \\ 0 & C_{62} & 0 & C_{64} & 0 & C_{66} \end{bmatrix}$$

We can permute the columns and rows of this matrix an even number of times to obtain the block matrix

$$\begin{bmatrix} C_{11} & C_{13} & C_{15} & 0 & 0 & 0 \\ C_{31} & C_{33} & C_{35} & 0 & 0 & 0 \\ C_{51} & C_{53} & C_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{22} & C_{24} & C_{26} \\ 0 & 0 & 0 & C_{24} & C_{44} & C_{46} \\ 0 & 0 & 0 & C_{26} & C_{46} & C_{66} \end{bmatrix}$$

Thus we observe the more computationally efficient approach to computing the eigenvalues by computing the eigenvalues of the block submatrices.

3.6. Five and Six Dimensions. First we consider curves in \mathbb{R}^5 with constant $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ which, up to translation, will have the form

$$\gamma(t) = (a \cos(\alpha t), a \sin(\alpha t), b \cos(\beta t), b \sin(\beta t), ct)$$

Letting $F_k = a^2 \alpha^k + b^2 \beta^k$, we obtained the following formulas:

$$1 = a^2 \alpha^2 + b^2 \beta^2 + c^2$$

$$\lambda_1 = \frac{1}{3} \epsilon^2 + O(\epsilon^4)$$

$$\lambda_2 = \frac{1}{20} F_4 \epsilon^4 + O(\epsilon^6)$$

$$\lambda_3 = \frac{1}{1575} (a^2 b^2 \alpha^2 \beta^2 (\alpha^2 - \beta^2)^2 + c^2 F_6) \epsilon^6 + O(\epsilon^8)$$

$$\lambda_4 = \frac{1}{63504} \frac{a^2 b^2 \alpha^4 \beta^4 (\alpha^2 - \beta^2)^2}{F_4} \epsilon^8 + O(\epsilon^{10})$$

$$\lambda_5 = \frac{1}{9823275} \frac{a^2 b^2 \alpha^6 \beta^6 (\alpha^2 - \beta^2)^2}{a^2 b^2 \alpha^2 \beta^2 (\alpha^2 - \beta^2)^2 + c^2 F_6} \epsilon^{10} + O(\epsilon^{12})$$

Using elimination theory, we establish the following representations of the κ_i in terms of the local singular values:

$$\kappa_1 = \frac{\sqrt{20}}{3} \frac{\sigma_2}{\sigma_1^2}, \quad \kappa_2 = \frac{\sqrt{105}}{2} \frac{\sigma_3}{\sigma_1 \sigma_2}, \quad \kappa_3 = \frac{\sqrt{336}}{5} \frac{\sigma_4}{\sigma_1 \sigma_3}, \quad \kappa_4 = \frac{\sqrt{825}}{4} \frac{\sigma_5}{\sigma_1 \sigma_4}$$

In a similar manner, for curves in \mathbb{R}^6 of the form

$$\gamma(t) = (a \cos(\alpha t), a \sin(\alpha t), b \cos(\beta t), b \sin(\beta t), c \cos(\delta t), c \sin(\delta t))$$

we obtain these same expressions for $\kappa_1, \kappa_2, \kappa_s, \kappa_4$ plus the additional relationship

$$\kappa_5 = \frac{\sqrt{1716}}{7} \frac{\sigma_6}{\sigma_1 \sigma_5}.$$

Throughout this section, we have assumed the curve to be parameterized with respect to arc length. The local computations can still be made without this assumption. What would change in the formulas in the previous section is that we would replace the assumption that $\|\gamma^{(1)}(t_0)\| = 1$ with $\|\gamma^{(1)}(t_0)\| = r$. We obtain the same connection between the higher curvature functions and *ratios* of singular values. We summarize the results of the previous pages in the following:

Theorem 3.2. *Let $\gamma : I \rightarrow \mathbb{R}^n$ be a parametric curve of class C^{n+1} , regular of order n with $n \leq 6$. Let $\kappa_i(t)$ denote the i^{th} curvature function of γ evaluated at t and let $\sigma_i(t)$ denote the i^{th} local singular value of γ at t . For each $t \in I$ and each $i < n$,*

$$\kappa_i(t) = \sqrt{a_i} \frac{\sigma_{i+1}(t)}{\sigma_1(t)\sigma_i(t)} \quad \text{with} \quad a_1 = \frac{20}{9}, a_2 = \frac{105}{4}, a_3 = \frac{336}{25}, a_4 = \frac{825}{16}, a_5 = \frac{1716}{49}$$

It is easy to check that the formula

$$a_{k-1} = \left(\frac{k}{k + (-1)^k} \right)^2 \frac{4k^2 - 1}{3}$$

is consistent with the first 5 values of a_i given above. Perhaps surprisingly, the numerator of this series arises in the number of Kekulé structures in benzenoid hydrocarbons [5] and the degrees of projections of rank loci [1]. We suspect Theorem 3.2 holds more generally. In particular, we make the following conjecture:

Conjecture 3.3. *Let $\gamma : I \rightarrow \mathbb{R}^n$ be a parametric curve of class C^{n+1} , regular of order n . Then for each $k \leq n$,*

$$\kappa_{k-1}(t) = \frac{k}{k + (-1)^k} \sqrt{\frac{4k^2 - 1}{3}} \frac{\sigma_k(t)}{\sigma_1(t)\sigma_{k-1}(t)}$$

Theorem 3.2 shows that the conjecture is true for $\kappa_1, \dots, \kappa_5$. We have numerically verified the conjecture for $\kappa_6, \kappa_7, \kappa_8$. This was done by generating curves with prescribed non-constant curvature and solving the system $E' = EK$ numerically. Then, the local singular values were numerically approximated from the numerically generated curves.

4. AN EXAMPLE

We consider the twisted cubic curve in \mathbb{R}^3 given parametrically as $\gamma(t) = [t, t^2, t^3]$. The Frenet-Serret frame can be shown to be:

$$e_1(t) = \begin{bmatrix} \frac{1}{\sqrt{1+4t^2+9t^4}} \\ \frac{2t}{\sqrt{1+4t^2+9t^4}} \\ \frac{3t^2}{\sqrt{1+4t^2+9t^4}} \end{bmatrix} \quad e_2(t) = \begin{bmatrix} \frac{t(2+9t^2)}{\sqrt{1+4t^2+9t^4}\sqrt{1+9t^2+9t^4}} \\ \frac{1-9t^4}{\sqrt{1+4t^2+9t^4}\sqrt{1+9t^2+9t^4}} \\ \frac{3t+6t^3}{\sqrt{1+4t^2+9t^4}\sqrt{1+9t^2+9t^4}} \end{bmatrix} \quad e_3(t) = \begin{bmatrix} \frac{3t^2}{\sqrt{1+9t^2+9t^4}} \\ \frac{-3t}{\sqrt{1+9t^2+9t^4}} \\ \frac{1}{\sqrt{1+9t^2+9t^4}} \end{bmatrix}$$

while the functions $\kappa_1(t), \kappa_2(t)$ can be shown to be

$$\kappa_1(t) = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}} \quad \kappa_2(t) = \frac{3}{1+9t^2+9t^4}$$

Let $\epsilon = .001$ and let $t = 3$. If we consider the singular value decomposition $C_\epsilon(t) = U_\epsilon(t)\Sigma_\epsilon(t)U_\epsilon^T(t)$ for $\gamma(t)$ then we can consider the singular vectors of $C_\epsilon(t)$ as a proxy for the local singular vectors of $\gamma(t)$ at $t = 3$ and compare to the exact value for $e_i(t)$ at $t = 3$. For instance, comparing the first singular vector to the first frame vector, we get

$$u_{1,\epsilon}(3) = \begin{bmatrix} .036131465 \\ .216788800 \\ .975549656 \end{bmatrix} \quad e_1(3) = \begin{bmatrix} .036131468 \\ .216788812 \\ .975549654 \end{bmatrix}$$

The other singular vectors, $u_{2,\epsilon}(3), u_{3,\epsilon}(3)$ are similarly close to $e_2(3), e_3(3)$. If we consider

$$\sqrt{a_i} \frac{\sqrt{\lambda_{i+1,\epsilon}(t)}}{\sqrt{\lambda_{1,\epsilon}(t)}\sqrt{\lambda_{i,\epsilon}(t)}} \text{ as a proxy for } \kappa_i = \sqrt{a_i} \frac{\sigma_{i+1}(t)}{\sigma_1(t)\sigma_i(t)}$$

then we obtain the following estimates:

$$\kappa_1(3) \approx .0026865640, \quad \kappa_2(3) \approx .0036991369$$

whereas using the exact formulas, we can compare these values to

$$\kappa_1(3) = .0026865644..., \quad \kappa_2(3) = .0036991368...$$

For these approximations, we used $\epsilon = 10^{-3}$. With a choice of $\epsilon = 10^{-6}$, for this example, we observed about 13 digits of accuracy. This example illustrates how the theorems of the previous section can be used to obtain very good approximations of both the Frenet-Serret frame and values of the curvature functions by considering small values of ϵ .

5. CONCLUSION

In this paper, we established the close connection between the Frenet-Serret apparatus and the local singular value decomposition of regular curves in \mathbb{R}^n . The local singular value decomposition was defined as the limit of the singular value decomposition of a family of covariance matrices defined on the curve. In particular, we showed in Theorem 3.1 that the Frenet-Serret frame and the local singular vectors of regular curves in \mathbb{R}^n agree (up to a factor of ± 1). In addition we showed in Theorem 3.2 that values of each of the curvature functions can be expressed in terms of ratios of local singular values for regular curves in \mathbb{R}^n with $n \leq 6$. Conjecture 3.3 concerns an extension of Theorem 3.2 to arbitrary dimension. We have numerically checked this conjecture for curves lying in \mathbb{R}^n with $n \leq 9$. The techniques also allow for highly accurate approximations for the Frenet-Serret apparatus.

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DEPARTMENT OF MATHEMATICS

Current address: Colorado State University

E-mail address: `arn@math.colostate.edu`

DEPARTMENT OF COMPUTER SCIENCE

Current address: Colorado State University

E-mail address: `draper@cs.colostate.edu`

DEPARTMENT OF MATHEMATICS

Current address: Colorado State University

E-mail address: `kirby@math.colostate.edu`

DEPARTMENT OF MATHEMATICS

Current address: Colorado State University

E-mail address: `peterston@math.colostate.edu`